UNIT-II

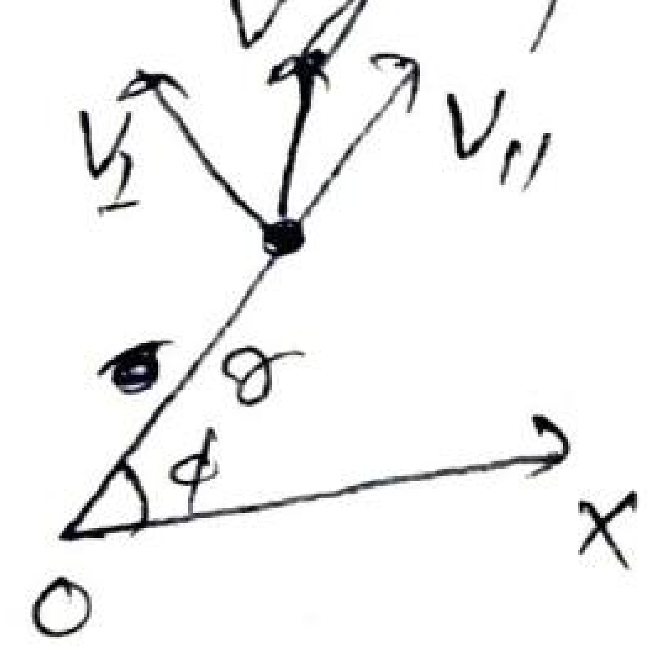
Angular velocity :- Angular velocity refers to how fast an object rotates or revolves relative to another point.

There are two types of angular velocity

(i) Orbital angular velocity

(ii) Spin angular velocity

In general, angular velocity is measured in angle per unit time.



$$\omega = \frac{d\phi}{dt}$$

If ϕ is measured in radians, the arc length from the positive x-axis around the circle to the particle is $l = r\phi$ and the linear velocity is $v = \frac{dl}{dt} = r\omega(t) \Rightarrow \omega = \frac{v}{r}$

Angular momentum :-

The angular momentum of a particle about a fixed in an interstial frame is the moment of its linear momentum about that point. That is why it is called moment of momentum. It is measured by the product of the linear momentum $\vec{p}$ of the particle and its vector distance $\vec{r}$ from the fixed or the reference point.

$$\vec{J} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v})$$

Torque :- The torque or the time rate of change of angular momentum of the system about the fixed or the reference point.

$$\frac{d\vec{J}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}) = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

Now, $\frac{d\vec{r}}{dt} \times \vec{p} = \vec{v} \times m\vec{v} = 0$

$$\frac{d\vec{J}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt} = \vec{r} \times \vec{F}$$

The vector product $\vec{r} \times \vec{F}$ is called torque.

Conservation of angular momentum:-

When the external torque (or the sum of external torques) applied to a system of particles is zero, the total angular momentum of the system remains conserved.

$$\vec{\tau} = \frac{d\vec{J}}{dt} = \vec{r} \times \vec{F} = 0$$

$$\frac{d\vec{J}}{dt} = 0 \Rightarrow \vec{J} = \text{Constant.}$$

Equation of motion:-

Equations of motion are equations that describe the behaviour of a physical system in terms of its motion as a function of time.

There are two ~~two~~ main descriptions of motion dynamics and kinematics. Dynamics is general, since the momenta, forces and energy of particles. ~~Kinematics is simpler~~ Kinematics is simpler. It concerns only variable derived from the positions of object and time.

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Moment of inertia :-

The inability of a body to change by itself its state of rest or uniform along a straight line is an inherent property of matter and is called inertia. The greater the mass of body, the greater the resistance offered it to any change in its state of rest or linear motion.

The nature of moment of a couple, for only a couple can oppose another couple. Hence the name moment of inertia given to it.

~~The moment of inertia of a body about a fixed axis of rotation may also be defined as the torque that~~

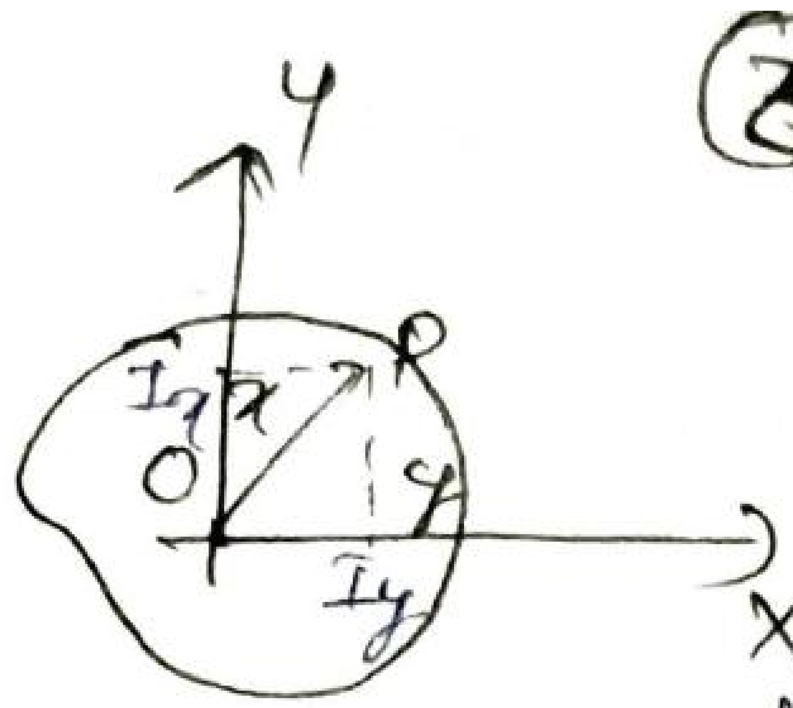
The moment of inertia of a rigid body about a given axis of rotation as the sum of products of masses of various particles of the body and squares of their respective distances from the axis.

$$I = \Sigma m r^2$$

General Thearm of Moment of Inertia :-

Thearm of perpendicular axes :-

(a) For a plane laminar body :- The thearm states that the moment of inertia of a plane lamina about an axis perpendicular to the plane of the lamina is equal to the sum of the moments of inertia of the lamina about two axes perpendicular to each other, in its own plane and intersecting each other at the point where the perpendicular axis passes through it.



$I = I_x + I_y$

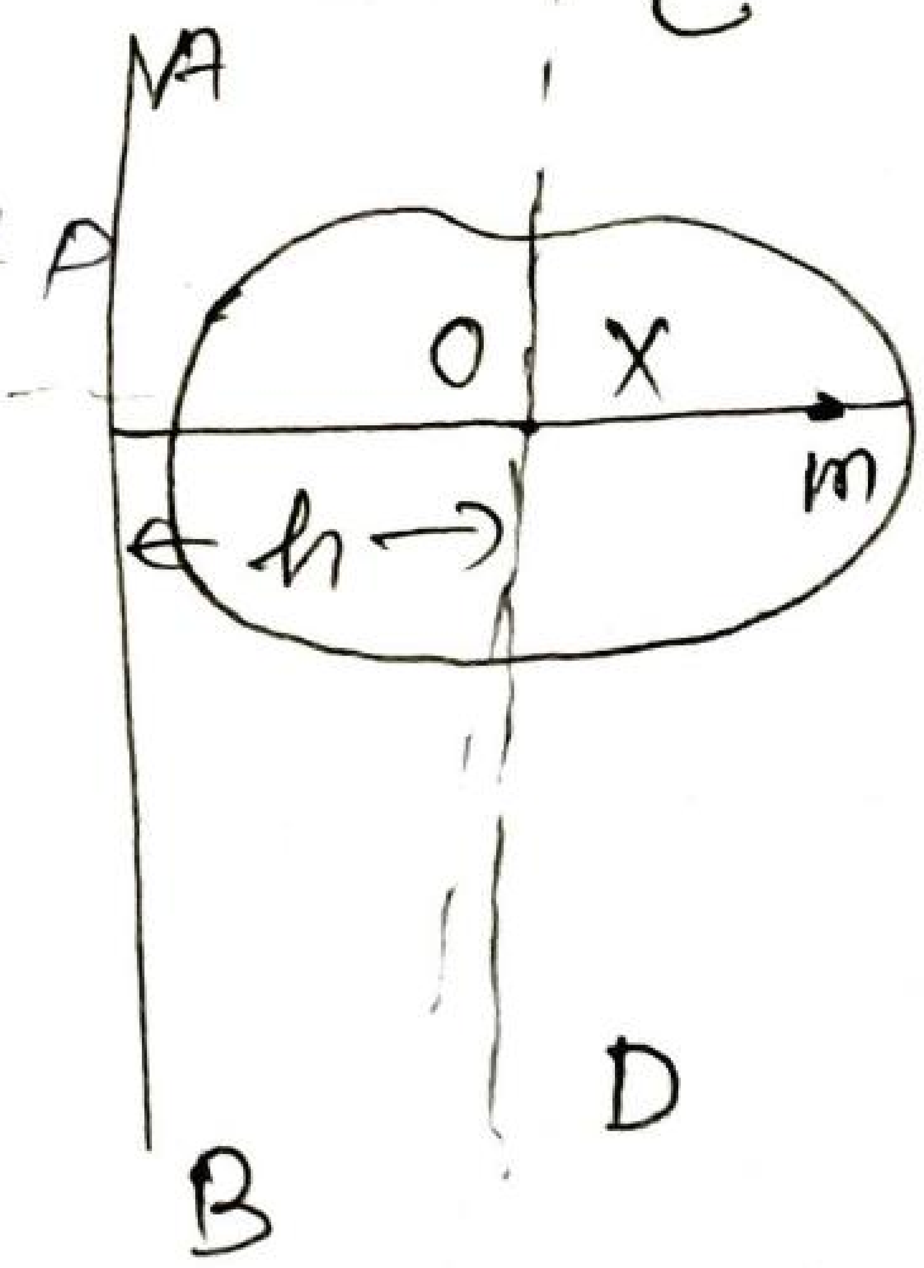
For considering a particle of mass m at P distant r from O and x and y from the axes OY and OX respectively, we have

$$I = \Sigma m r^2, \; I_x = \Sigma m y^2 \text{ and } I_y = \Sigma m x^2$$

$$I_x + I_y = \Sigma m y^2 + \Sigma m x^2 = \Sigma m (y^2 + x^2)$$
$$= \Sigma m r^2$$

$$\boxed{I = I_x + I_y}$$

Theorm of parallel axes :- The theorm state that the moment of inertia of a body about any axis is equal to its moment of iner about a parallel axis through its centre of mass, of the body and plus the product of the mass of the body and square of the distance between the two axes.



Consider a small element of mass m of the lamina at the point P distance x from CD, we have moment of inertia of the element about the axes $AB = m(x+h)^2$

Therefore, moment of inertia of whole lamina about the axis AB i.e $I = \Sigma m(x+h)^2$

$$I = \Sigma m x^2 + \Sigma m h^2 + 2\Sigma m x h$$

we know that $\Sigma m x^2 = I_{c.m.}$

$$I = I_{cm} + \Sigma m h^2 + 2\Sigma m x h$$

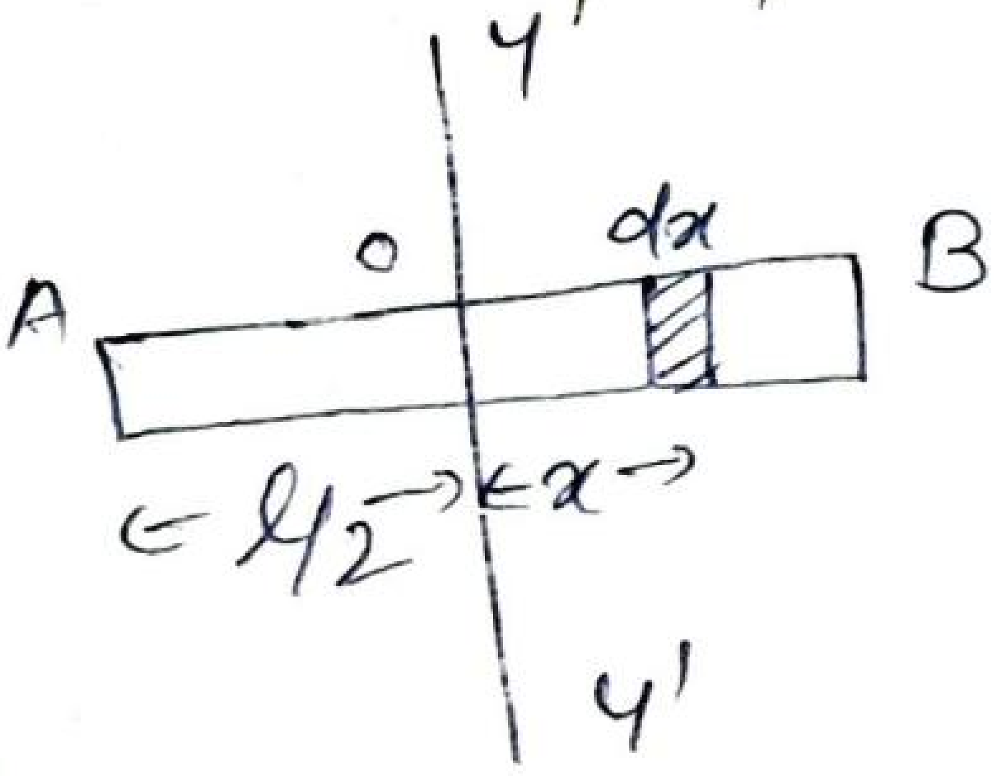
Now $\Sigma m h^2 = h^2 \Sigma m = M h^2$, where M is the mass of whole lamina, $\Sigma m x = 0$

So $\cancel{I = I_{c.m.}}$

$$\boxed{I = I_{c.m} + M h^2}$$

Moment of inertia of a uniform rod:-

(i) About an axis through its centre and perpendicular to its length:-



Let AB be a thin uniform rod of length l and mass M free to rotate an axis YOY' passing through its centre O

Rod is uniform, its mass per unit length is $= M/l$

Consider a small element of the rod of length dx at a distance x.

Mass of the element $= \left(\frac{M}{l}\right) dx$

M.I about the axis (YOY') through $O = \left(\frac{M}{l}\right) x^2 dx$

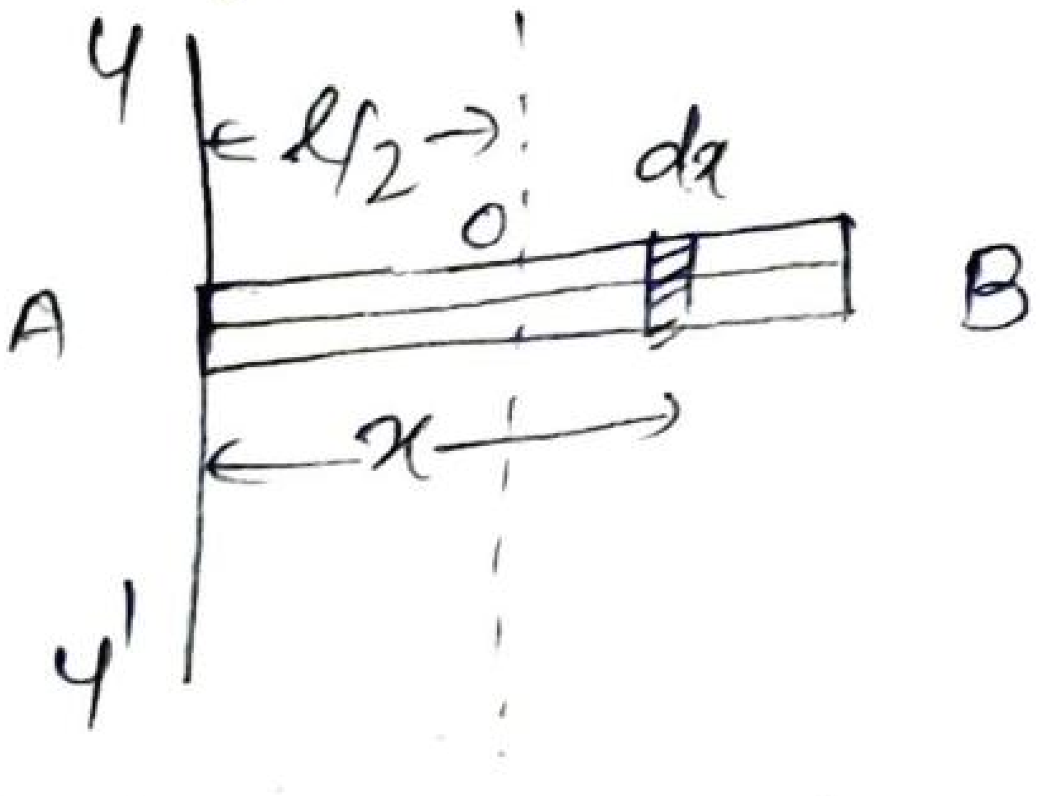
M.I of the whole rod about the axis YOY' is

$$I = 2\int_0^{l/2} \frac{M}{l} x^2 dx = \frac{2M}{l} \cdot \frac{x^3}{3}\Big|_0^{l/2} = \frac{2M}{l} \cdot \frac{l^3}{24}$$

$$= \frac{Ml^2}{12}$$

(ii) About an axis through one end of the rod and perpendicular to its length:-

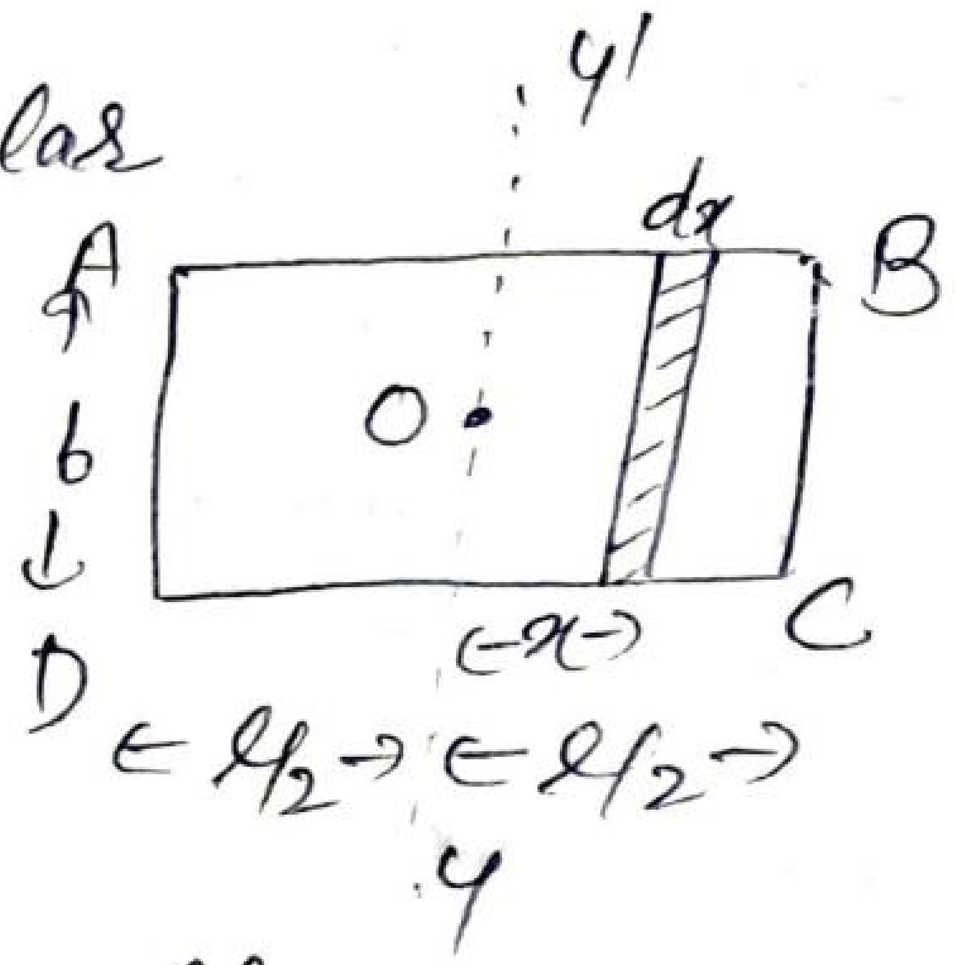
$$I = \int_0^l \frac{M}{l} x^2 dx = \frac{M}{l} \frac{x^3}{3}\Big|_0^l$$



$$I = \frac{Ml^2}{3}$$

(2) Moment of inertia of a rectangular lamina (or bar):

(i) About an axis through its centre and parallel to one side:- Let ABCD be a rectangular lamina of length l and breadth b and mass M and let YOY' be the axis through its centre O.



Consider an element or a small rectangular strip of the lamina, parallel to and at a distance x from the axis. The area of the strip is $= dx \times b$

Mass per unit area of lamina $= \dfrac{M}{(l \times b)}$

Mass of the strip is $= \dfrac{M}{(l \times b)} \times dx \times b = \dfrac{M}{l} dx$

M.I of the element about the axis YOY'

$$= \frac{M}{l} dx \cdot x^2$$

$$I = \int_{-l/2}^{+l/2} \frac{M}{l} dx \cdot x^2 \Rightarrow 2\int_0^{l/2} \frac{M}{l} x^2 dx$$

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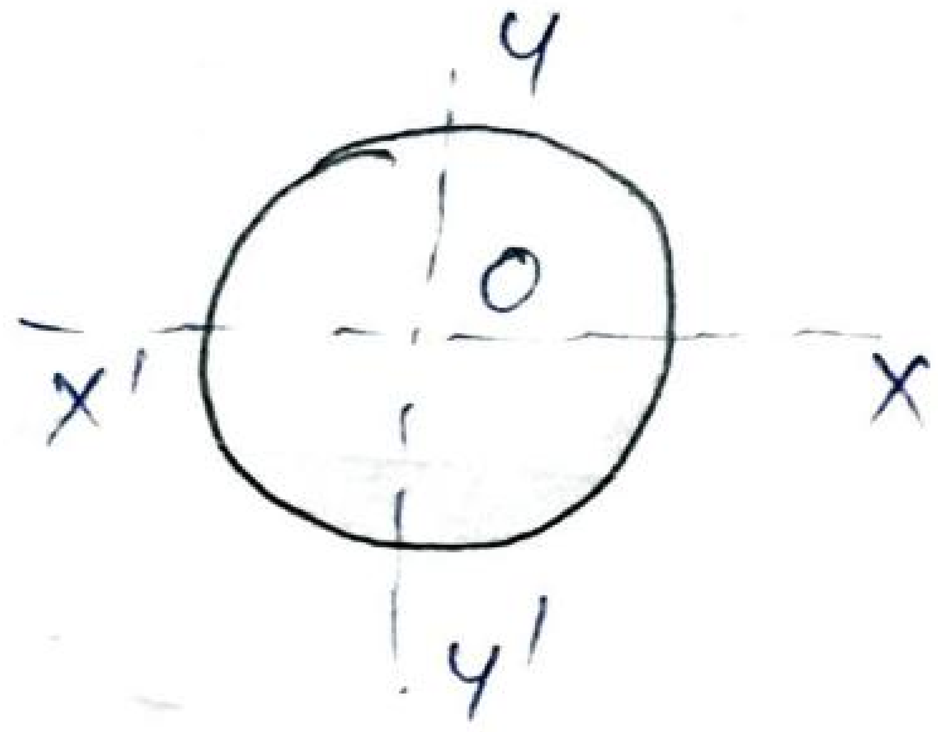
Moment of inertia of a hoop or a thin circular ring:-

(i) About an axis through its centre and perpendicular to its plane:-



Let the radius of the hoop or thin circular ring be R and its mass M,

Consider a particle of mass m of a hoop or the ring.

M.I about an axis passing through the centre O of the hoop $= mR^2$

M.I. of the entire hoop or ring

$$I = \Sigma m R^2 = MR^2$$

(ii) About its diameter:-

Due to symmetry, the M.I. of the hoop or the ring will be same about one diameter as another.

$$I + I = MR^2 \Rightarrow 2I = MR^2$$

$$\Rightarrow I = \frac{MR^2}{2}$$

(4) Moment of inertia of a circular lamina or disc:-

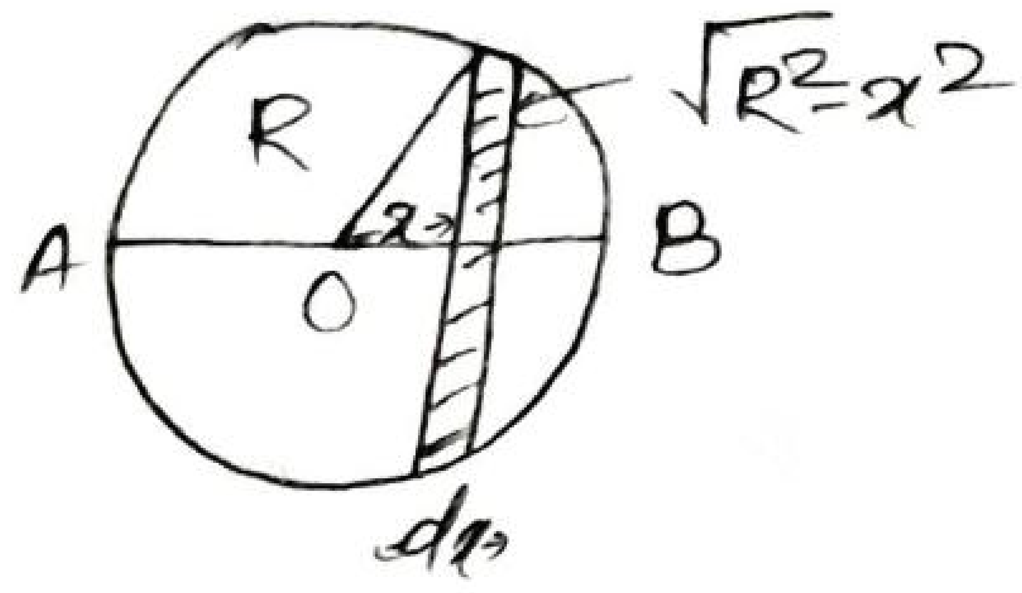
(i) About an axis through its centre and perpendicular to its plane:-

$$I = \frac{MR^2}{2}$$

(ii) About a diameter:-

$$I = \frac{MR^2}{4}$$

(10) Moment of inertia of a solid sphere:-



(i) About a diameter:-

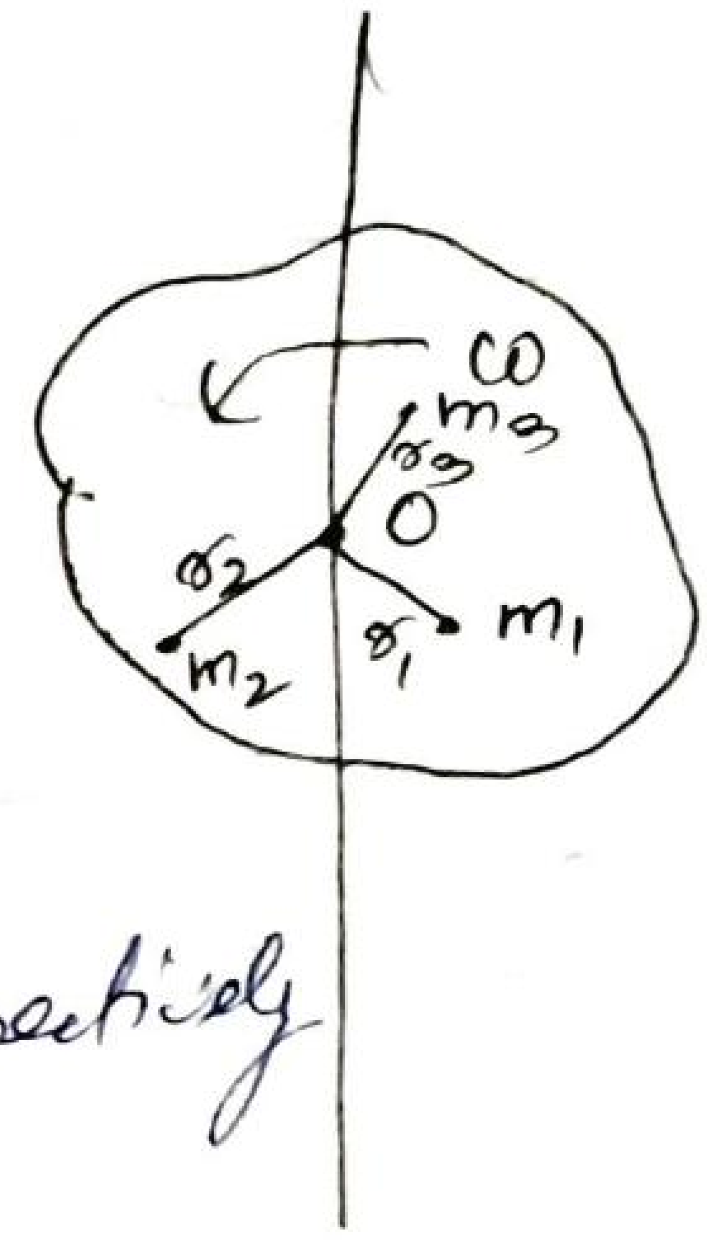
$$I = \frac{2}{5}MR^2$$

(ii) About a tangent:-

$$I = \frac{2}{5}MR^2 + MR^2 \Rightarrow \frac{7}{5}MR^2$$

Kinetic Energy of Rotation:-

(i) Kinetic energy of a body rotating about an axis through its centre of mass:-



The body is clearly made up of a large number of particles of masses $m_1, m_2, m_3 \ldots$ etc, at respectively distance $r_1, r_2, r_3 \ldots$ etc. From the axis AB through O. Since their angular velocity is the same (ω), their respective ~~kinetic energies~~ linear velocities equal to

$$v_1 = r_1\omega,\ v_2 = r_2\omega,\ v_3 = r_3\omega \ldots \text{etc.}$$

Hence kinetic energy

$$K_1 = \frac{1}{2}m_1v_1^2 = \frac{1}{2}m_1r_1^2\omega,\ K_2 = \frac{1}{2}m_2r_2^2\omega \ldots$$

Total K.E. of the body itself

$$K = \frac{1}{2}m_1\omega^2r_1^2 + \frac{1}{2}m_2r_2^2\omega^2 \ldots$$

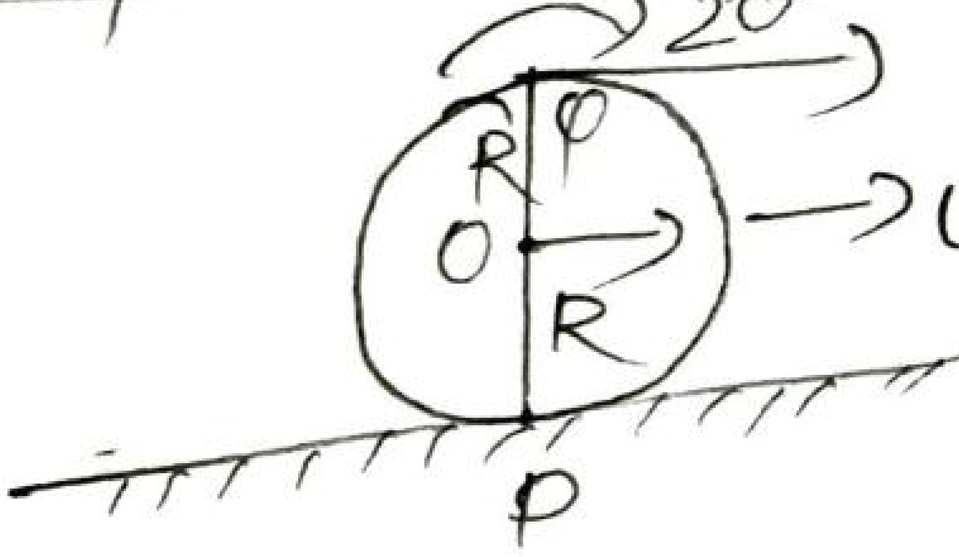
$$= \frac{1}{2}\omega^2(m_1 r_1^2 + m_2 r_2^2 + \dots)$$

$$= \frac{1}{2}\omega^2 \Sigma m r^2 = \frac{1}{2}\omega^2 MK^2 = \frac{1}{2}I\omega^2$$

~~so~~ where $\Sigma m r^2 = MK^2$ with M as the mass of the body and K, its radius of gyration about the axis of ~~root~~ rotation AB.

(ii) Kinetic energy of a rotating body whose centre of mass has also a linear velocity:-

(a) Case of a body rolling along a plane surface:-



If $I_{c.m}$ be the moment of inertia of body about a parallel axis through its centre of mass we have, by the principle of parallel axes $I_P = I_{c.m} + MR^2$

$$\text{K.E. of rolling body} = \frac{1}{2}(I_{c.m} + MR^2)\omega^2$$

$$= \frac{1}{2}I_{c.m}\omega^2 + \frac{1}{2}MR^2\omega^2$$

$$= \frac{1}{2}I_{c.m}\omega^2 + \frac{1}{2}Mv^2 \quad (\because v = r\omega)$$

where $I_{c.m} = MK^2$ $[\because \omega = v/R$

K.E of the rolling body

$$= \frac{1}{2}MK^2\frac{v^2}{R^2} + \frac{1}{2}Mv^2 = \frac{1}{2}Mv^2\left(\frac{K^2}{R^2} + 1\right)$$